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Predictability time from the seismic signal in an earthquake model

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Abstract. We compute the maximum Lyapunov exponent λ of an earthquake model which exhibits deterministic chaos and we discuss its relation with the predictability time of the system. A method is proposed to estimate λ by the calculation of the entropy of Markov processes which mimic (i) a Poincaré map of the model and (ii) a random map related to the seismic signal. The latter map can be obtained using experimental records generated by low-dimensional chaotic systems where Poincaré maps are not feasible.

Phenomena related to deformations and fractures of the earth crust are very complex. From a theoretical point of view, it is necessary to find a model as simple as possible which at the same time exhibits some qualitative feature of the earthquake dynamics. One of the first attempts was due to Burridge and Knopoff (1967). They have introduced a stick–slip model of coupled oscillators which mimics the interaction of two faults. In practice one considers blocks on a rough support connected to each other by springs. They are also connected by other springs to a driver that pulls the chain of the blocks at a very low constant velocity. The blocks stick until the pulling spring force overwhelms the static friction and then one or more blocks slide. After sliding (a sort of seismic event), the blocks stick again in new equilibrium positions until the next event. To simulate the earthquake dynamics, Carlson and Langer (1989) studied a chain with a large number of homogeneous blocks (as many as 200) which interact by equal springs connecting nearest neighbours. The dynamic friction is chosen to depend in a nonlinear way on the velocity; it decreases with increasing block speed. This model has been shown to exhibit power laws (Gutenberg and Richter 1956†, Omori 1894) which are a common mark of earthquake statistics. Moreover, it has been argued that the appearance of power laws in this deterministic system is connected to the strong intermittency in the chaotic behaviour due to the cooperative effect of many degrees of freedom (Crisanti *et al* 1992).

On the other hand, a chaotic evolution (without power laws and with weak intermittency) also appears in a model with only two blocks (Huang and Turcotte 1990) (see also Narkounskaia *et al* 1992), where the friction acting on one block is larger than that on the second by a factor β. Although the model cannot describe the dynamics of the elastic deformations in a single homogeneous fault, it can be regarded as a good description of the dynamics of two coupled large segments of a fault. Huang and Turcotte used it to analyse the earthquake records at Wallace Creek and Pallet Creek (two sites near the San Andreas fault) as well as in the subduction zone of Nankai in Japan.

† The Gutenberg–Richter law is a power law for the seismic moment M. However, one usually considers the probability of occurrence of an earthquake of magnitude $m \sim \ln M$, so that the law is formulated as $P(m) = A\mathrm{e}^{-bm}$ where $b \approx 1$.

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This paper discusses the importance of measuring the degree of chaos in such a system, since it allows one to relate the spatial inhomogeneity of an active zone (the factor β) to the degree of predictability of the seismic events originated by it. Our main result is a method which can be directly applied to signals of the seismic moment without reference to the evolution equations, which in general cases are very difficult to reconstruct with experimental signals.

For this purpose, we numerically compute the maximum Lyapunov exponent λ, which gives the mean rate of divergence of two initially close trajectories. In practice, even when the deterministic evolution equations are exactly known, the information obtained by a measure of the state of the system (here positions and speeds of the two blocks) is quickly lost in the chaotic regime. Indeed, when the measure precision is ϵ, after a time t the state of the system can be predicted with an incertitude of order $\epsilon \exp(\lambda t)$. It follows that the deterministic nature of the evolution equations are useful for a forecasting only up to a predictability time

$$T_{\mathrm{P}} = \lambda^{-1} \ln(A/\epsilon) \tag{1}$$

where A is the minimum precision which can be accepted to individuate at least some qualitative feature of the system. A sensible choice of A is a subjective matter, since it depends on the type of information requested. However, the crucial point is that the predictability time depends on the measure precision ϵ and on the toleration parameter A in an extremely weak way which can be safely ignored. For all purposes, the logarithm in (1) can be considered a constant of order unity, and the predictability time identified with the inverse Lyapunov exponent. The main difficulty stems from our ignorance of the evolution equations ruling the dynamics even of simple fault systems as those described by two block models. One needs a method to estimate the degree of chaos and thus the predictability time by an analysis of physical quantities such as the energy released in a seismic event. We have thus studied the simplest realistic model as a first step toward this goal. Our procedure is the following:

(i) We compute the Lyapunov exponent of the Huang–Turcotte model.

(ii) We find appropriate random processes which mimic the dynamics of a one-dimensional Poincaré map of the flow. The entropy of a two-step Markov process is a very precise estimate of the Lyapunov exponent.

(iii) We use the model to generate an 'experimental' seismic signal. Its time record is used to construct a sort of random mapping which can be described in terms of a Markov process. As there is no randomness in the starting model, this mapping is not the most accurate way to investigate the dynamical process of the model. However, in the analysis of time records, it could be much easier to obtain a random mapping related to the seismic signal rather than first return deterministic maps. The Huang–Turcotte model is an ideal playground to test the efficiency of random maps.

Let us briefly define the stick–slip model. The two blocks of equal masses M are connected by a spring with Hook coefficient k_{c} and they are linked to the driver by springs with equal coefficient k. The driver moves at constant speed v_{d} and we choose the reference frame where the driver is at rest (a stuck block is thus characterized by $\dot{y} = v_{\mathrm{d}}$). The evolution equations are

$$\begin{aligned} M\ddot{y}_1 + (k+k_{\mathrm{c}})y_1 - k_{\mathrm{c}}y_2 &= F_1 \\ M\ddot{y}_2 + (k+k_{\mathrm{c}})y_2 - k_{\mathrm{c}}y_1 &= F_2 \end{aligned} \tag{2}$$

where y_1, y_2 are the displacement from the equilibrium position, and F_1, F_2 are the friction forces. A block can be stuck since the static friction exactly balances the harmonic forces up to a threshold value. When the spring forces acting on the first (second) block are larger than F_0 (βF_0), the block slips and is slowed down by a dynamical friction with a velocity weakening form. It is convenient to define adimensional variables by introducing a reference speed v_f (an appropriate choice is the average slip speed) so that $Y_i = y_i k/F_0$, $\alpha = k_c/k$, $\beta = F_2/F_1$, $\gamma = F_0/(v_f\sqrt{Mk})$, $\tau = t\sqrt{k/M}$. We consider a dynamical friction which is slightly modified with respect to the Huang–Turcotte form:

$$F(\dot{Y}) = F_0/(1+\gamma|\dot{Y}-\nu|) \tag{3}$$

where $\nu = v_d/v_f$ is the drift speed v_d in adimensional units. The equations during a slip thus become

$$\begin{aligned}\ddot{Y}_1 + Y_1 + \alpha(Y_1 - Y_2) &= 1/(1+\gamma|\dot{Y}_1 - \nu|)\\ \ddot{Y}_2 + Y_2 + \alpha(Y_2 - Y_1) &= \beta/(1+\gamma|\dot{Y}_2 - \nu|)\end{aligned} \tag{4}$$

while when one of the two blocks sticks, one has in our reference frame

$$\ddot{Y}_1 = 0 \qquad \dot{Y}_1 = \nu \tag{5a}$$

or

$$\ddot{Y}_2 = 0 \qquad \dot{Y}_2 = \nu \tag{5b}$$

respectively if $|Y_1 + \alpha(Y_1 - Y_2)| < 1$, or $|Y_2 + \alpha(Y_2 - Y_1)| < \beta$. The quantity $T_\nu = \nu^{-1}$ is the natural (adimensional) time unit of the system. Realistic ν values are of order 10^{-8} or less, that is the existing ratio between the typical duration of a seismic event and the average recurrence time (Carlson and Langer 1989). However, the main qualitative results are independent of ν and we have integrated the equation with $\nu = 0.01$. Following Huang and Turcotte (1990), we have fixed $\alpha = 1.2$ and $\gamma = 3$, and varied the friction ratio β. The evolution has been studied by the Poincaré map given by the intersection of the orbit with the plane $(\dot{Y}_2 - \dot{Y}_1, Y_2 - Y_1)$. At $\beta = 1$, the Poincaré map has a fixed point which, however, depends on the particular initial condition chosen. At increasing β, one observes a transition to chaotic behaviour via period-doubling bifurcations, as shown in figure 1. In other terms, for $\beta - 1$ small enough, the behaviour is periodic. The critical β for the transition to chaos depends on the form of the friction law but it is not sensitive to the initial conditions. In the chaotic regime, the Lyapunov exponent $\lambda_m(\beta)$ of the Poincaré map is positive and is related to the maximum Lyapunov exponent λ of the original flow by the relation

$$\lambda = \lambda_m/\langle\tau\rangle \tag{6}$$

where $\langle\tau\rangle$ is the average time interval between two successive intersections of the flow with the Poincaré section. We have estimated the rate of divergence of nearby trajectory by a numerical integration of (4). In figure 2, the resulting Lyapunov exponent λ is shown as function of β. Note that the Lyapunov exponent λ has the dimension of an inverse of a time. In this paper we use the natural time unit which corresponds to ν^{-1} integration steps.

One sees an alternation of chaotic and regular windows, while the envelope decreases nearly exponentially for large β. This indicates that, for strongly asymmetric frictions, the system becomes more and more regular since the activity tends to concentrate on the less

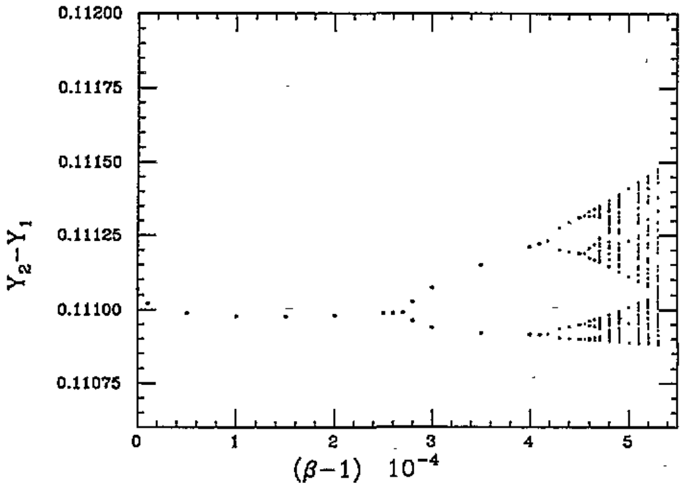

Figure 1. Period-doubling transition to chaos in the slip–stick model with $\alpha = 1.2$, $\gamma = 3$ and $\nu = 0.01$: asymptotic values of $Y_2 - Y_1$ at varying β.

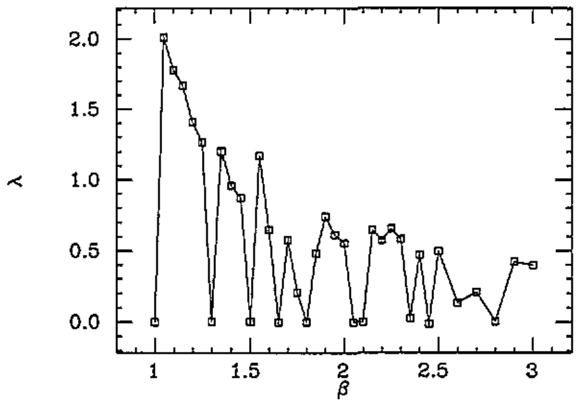

Figure 2. Maximum Lyapunov exponent λ in natural units as a function of β (the ratio between the frictions of the two blocks) with $\alpha = 1.2$, $\gamma = 3$ and $\nu = 0.01$. λ is estimated by the rate of exponential divergence of nearby trajectories obtained by a numerical integration of 4.

constrained block. In order to study the predictability problem, we have analysed in some detail the case $\beta = 2$. The Poincaré map in figure 3 is the set of the intersections of the representative point in the phase space $(Y_1, Y_2, \dot{Y}_1, \dot{Y}_2)$, with the plane $(\dot{Y}_2 - \dot{Y}_1, Y_2 - Y_1)$. Plotting $Y_2 - Y_1$ respectively before and after a slip we have obtained the one-dimensional map $x_{n+1} = f(x_n)$ with $x \in I$, $I \subset R$, $n \in N$, where $f : I \to I$. The numerical points can be fitted by the polynomial forms $f_K(x) = \sum_i a_i x^i$ defined on appropriate subintervals I_K of I. The Lyapunov exponent is easily obtained by the tangent map

$$z_{n+1} = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_n} z_n \tag{7}$$

where $z \in R$ is the tangent vector which should be regarded as an infinitesimal perturbation on the map trajectory x_n. One thus has

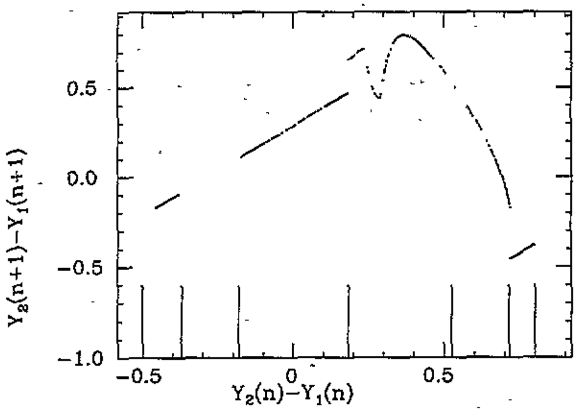

Figure 3. Poincaré map $Y_2(n+1) - Y_1(n+1)$ against $Y_2(n) - Y_1(n)$ of the flow generated by equations (4) where the parameters are $\beta = 2$, $\alpha = 1.2$ and $\gamma = 3$. The intervals of the partition are indicated by vertical bars on the x axis (note that the second interval from the left does not belong to the invariant set of the map and is not a partition element).

$$\lambda_m = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{\|z_n\|}{\|z_0\|}\right). \tag{8}$$

In our case, after 10^5 iterations of the tangent map (7), one gets

$$\lambda_m = 0.350 \pm 0.005.$$

The average time delay between two successive intersections of the flow with the Poincaré map is in natural time units

$$\langle\tau\rangle = 0.640 \pm 0.005$$

so that the maximum Lyapunov exponent for the original flow is

$$\lambda = 0.55 \pm 0.01.$$

We have also computed λ directly from the numerical integration of the original differential equations, by considering the the rate of divergence of two initially nearby orbits. After integrating for 1000 natural time units, we find

$$\lambda = 0.55 \pm 0.05$$

in good agreement with the more precise estimate obtained via (8). We have also computed the intermittency degree (Paladin and Vulpiani 1987). The system exhibits the standard type of behaviour of low-dimensional chaotic systems. There are small fluctuations in the effective Lyapunov exponents $\gamma_\tau(n) = (1/\tau)\ln(|z_{n+\tau}|/|z_n|)$ computed at finite time τ around the mean value λ, and the distribution can be well approximated by a Gaussian for small $|\gamma - \lambda|$, though it is characterized by a entropy function $S(\gamma)$ which has no universal form and depends on the particular dynamical system considered (see for a discussion Paladin and Vulpiani (1987)). It is worth stressing that this is not the case for the Carlson and Langer model when there are $N \geqslant 100$ blocks. In fact, for $N \geqslant 100$, the variance of the

γ distribution does not tend to a finite value but diverges as τ^w for a time delay $\tau \to \infty$, indicating that $S(\gamma)$ does not exist and that there is no characteristic time scale because the intermittency is too strong. This is conjectured to be the very origin of the Gutenberg–Richter law in many-block models (Crisanti *et al* 1992). On the other hand, the Huang and Turcotte model has negligible intermittency. It follows that there are small fluctuations of the predictability time around its mean value, which can can be estimated as

$$T_P \approx \lambda^{-1} = 1.8 \pm 0.2$$

in natural time units. This means that knowledge of the initial state of the system allows one to give sensible predictions only for times smaller than twice the typical time scale, proportional to the inverse of the driving speed ν. This time scale can be shown (Carlson and Langer 1989) to be of the order of the 'loading' time, that is the average time interval between two large seismic events.

In view of applications to signal analysis, it is interesting to estimate the Lyapunov exponent of the Poincaré map in terms of appropriate random processes (Crisanti *et al* 1989). We construct a partition of the invariant set under the map dynamics into M intervals. Successive iterations of the map produce a time sequence of intervals visited by the system, and thus a symbolic dynamics of M symbols. In a chaotic regime, the statistical properties of the deterministic symbolic dynamics and of an appropriate random sequence of symbols should be the same. As a first approximation we can describe the dynamics by a first-order Markov chain. Therefore, we estimate

(i) the stationary probability vector p whose elements p_i are given by the visit frequence of the interval i; and

(ii) the $M \times M$ transition matrix W_{ij} by a numerical computation of the probability that the system jumps from the interval i to the interval j in one time step.

It is worth recalling that p is the left eigenvector of the transition matrix corresponding to the largest eigenvalue 1. The entropy of the Markov chain is defined as

$$S_1 = -\sum_{i,j=1}^{M} p_i W_{ij} \ln(W_{ij}). \tag{9}$$

This entropy is expected to tend to the Kolmogorov–Sinai K entropy (Eckmann and Ruelle 1985) of the dynamical system when the number of the partition elements $M \to \infty$. In general, K is given by the sum of the positive Lyapunov exponent and for a one-dimensional expanding map, it coincides with the maximum Lyapunov exponent. Choosing a partition of $M = 5$ intervals (see figure 3), we have found $S_1 = 0.54$ by a diagonalization of the transition matrix $W_{i,j}$ which has been estimated by a numerical iteration of the polynomial form approximating the one-dimensional map of figure 3. In the appendix, we give the results for different partitions of the maps. Let us remark that increasing the number of partition elements does not substantially improve the estimate obtained with $M = 5$. In practice, it is more efficient to consider higher-order Markov process at fixed M. In this case, the transition operator is a tensor. For instance we have considered (at $M = 5$) a second-order Markov chain and the corresponding tensor Q_{ilj} given by the probability that in two iterations the system jumps from the interval i to l and then from l to j. The entropy of this process is

$$S_2 = -\sum_{i,l,j=1}^{M} p_i W_{il} Q_{ilj} \ln(Q_{ilj}) \tag{10}$$

where p_i is the stationary probability and W_{ij} is the one-step transition matrix, previously defined. The numerical calculation gives $S_2 = 0.36$ which is an accurate estimate of the Lyapunov exponent $\lambda_m = 0.350$ given by the integration of the tangent map (7). A moment of reflection shows that increasing the correlation of the random process (i.e. the order n of the Markov chain) should lower the corresponding entropy value, i.e. should increase the predictability time (estimated by S_n^{-1}). Once fixed the partition elements, one can determine their frequency visit and the transition probabilities by a numerical iteration of the deterministic mapping. One can thus obtain a sequence of Markov processes of order n, all of them approximating the same chaotic dynamics. It is evident that the Bernoulli process ($n = 0$) is more unpredictable than a first-order Markov process, and so on. It follows that S_n^{-1} should be a non-decreasing function of n.

Our result shows that it is possible to estimate the Lyapunov exponent in a simple way without knowing either the differential equations which rule the system or the optimal partitions (the Markov partition or a generating partition (Eckmann and Ruelle1985)). However, our procedure is not feasible in generic situations of seismic interest since it is hard to obtain a well shaped one-dimensional map from a time signal of natural physical quantities, such as the energy released in an earthquake. In Burridge–Knopoff models, the seismic moment (proportional to the released energy) is the sum of the sliding runs during a single seismic event, that is

$$M_n = \sum_{i=1}^{2} Y_i(n+1) - Y_i(n) \tag{11}$$

where $Y_i(n)$ is the position of the ith block before the nth slip. As shown in figure 4, the map M_{n+1} against M_n of the seismic moment computed at subsequent events is multi-valued on the definition domain. This is a general feature which must be taken into account when analysing realistic signals generated from dynamical systems exhibiting low-dimensional chaos. As for the previous map, it is simple to find polynomial forms which fit the numerical points. To study the map, we have introduced a partition of $M = 9$ elements, as shown in figure 4. Since some points have more than one image, the simplest description of the dynamics is through a random map where a weight is assigned to each possible option. To simplify the numerical simulation, we have assumed that these weights are uniform on an element of the partition. Although the random assumption is only a hypothesis, using the previous rough approximations, we can easily compute the entropies of the corresponding one-step and two-step Markov processes:

$$S_1 = S_2 = 0.6$$

with an error of one per cent. We expect that these entropies should be larger than the Kolmogorov–Sinai entropy of the deterministic map. In this case, the predictability time is underestimated by a factor two. The fact is probably due to the lack of precision in defining the random mapping. Assigning the respective weights of different branches of the map as a function of M_n, and not of the partition interval, might improve the result. However, there is a good qualitative agreement between the Lyapunov exponent and the entropy of a Markov process.

In conclusion, the predictability of two-block models with asymmetric friction has severe limitations in the presence of chaotic behaviour. We have shown that the order of magnitude of the predictability time can be estimated by appropriate random mappings extracted by a time signal, without knowing the underlying deterministic evolution equation. Strictly

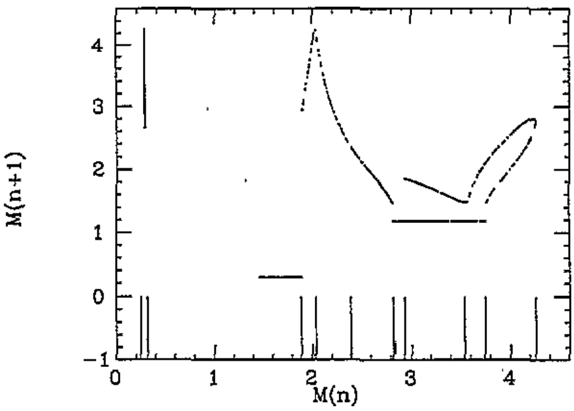

Figure 4. Multi-valued map of the seismic moments: M_{n+1} against M_n. The system parameters are the same as in figure 2. The intervals of the partition are indicated by vertical bars on the x axis.

speaking, our estimate is an upper bound of the inverse Lyapunov exponent but we think that the entropy of the Markov process associated with the random map is a much more sensible definition of the inverse predictability time. In real situations, the seismic moment is the only quantity accessible to experimental measurements and no first return map can be obtained.

Random maps such as the map of figure 4 could be interesting for other phenomena of geophysical interest which exhibit low-dimensional deterministic chaos, since our method works even when the evolution equations ruling the system are unknown. This is the most important result of this paper which, in principle, can be applied to different types of experimental signal.

Acknowledgments

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Appendix

In this appendix we give the result for the entropies S_1 and S_2 of the Markov processes associated with different partitions of the invariant set of the one-dimensional map shown in figure 3. We have considered partitions with 5, 8 and 13 elements.

The simplest partition with $M = 5$ elements can be found in figure 3.

The partition with $M = 8$ elements is obtained by individuating discontinuities, maxima and minima of the map. It follows that the third interval of the partition of figure 3 is split into the three intervals [0.184, 0.24], [0.24, 0.285], [0.285, 0.370].

The $M = 13$ elements of the last partition are given by the images of the first intervals of the partition in figure 3 under subsequent iterations of the mapping. This procedure does not cover the sub-interval [0.370, 0.390] of the invariant set. The first interval is [−0.5, −0.37], and the interval $[-0.18, 0.8]$ is divided into 12 elements with bounds in the points -0.18, -0.08, -0.105, 0.184, 0.24, 0.285, 0.370, 0.390, 0.470, 0.52638, 0.64, 0.7150189, 0.8.

The entropy values are respectively

$$S_1 = 0.54 \qquad S_2 = 0.36 \qquad \text{for } M = 5 \tag{A1}$$

$$S_1 = 0.72 \qquad S_2 = 0.39 \qquad \text{for } M = 8 \tag{A2}$$

$$S_1 = 0.48 \qquad S_2 = 0.38 \qquad \text{for } M = 13. \tag{A3}$$

The Kolmogorov–Sinai entropy of the natural probability measure (Eckmann and Ruelle 1985) has been estimated to be $K = 0.36$ by a numerical calculation of the Lyapunov exponent.

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